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Approximation of Unbounded Functions and Applications to Representations of Semigroups

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The purpose of this paper is to study the approximation of functions in m variables and its application to semigroup representation. First, two Bohman-Korovkin-type theorems are established for the respective approximations of unbounded, operator-valued and real-valued functions with noncompact supports in $\mathbb{R}^m$. Then we investigate several approximation operators: some of them are generalizations (to m dimensions) of well-known linear positive operators and some are apparently new. Finally, through these operators, the first approximation theorem provides a unified approach to a whole set of representation formulas for m-parameter (C_0)-semigroups of operators: special cases include well-known formulas due to Hille, Phillips, Widder, Kendall and Chung, as well as some new ones.

1. INTRODUCTION

The well known Bohman-Korovkin theorem [3, 14] states that if $\{L_n\}$ is a sequence of linear positive operators on $C[a, b]$, the space of real continuous functions on $[a, b]$, then $\|L_n f - f\|_\infty \to 0$ for every $f \in C[a, b]$, provided that this is true for $f(t) = 1$, $f(t) = t$ and $f(t) = t^2$. Efforts have been made by many authors to enlarge the domain of approximation operators to include bounded or unbounded functions with noncompact supports. It is worth while to mention here a few such versions. Müller [19] extended the theorem to functions which are bounded on $[a, \infty)$ and continuous on some $[c, d]$; Schurer [22] treated functions which are bounded on every finite interval and are of order $O(t^2)$ ($|t| \to \infty$); Ditzian's result [8] deals with functions satisfying the growth condition $|f(t)| \leq M(f)(t^2 + 1)\,\mu(t)$, $-\infty < t < \infty$, for some suitable function $\mu(t)$; Mamedov's theorem estimates the convergence rate of $(L_n f)(\xi)$ for functions f which are bounded on every finite interval, p times differentiable at ξ, and of order $O(|t|^p)$ ($|t| \to \infty$), where $p \geq 1$ (see [15] or [25]); Hsu [12] considered positive operators which approximate unbounded functions of order $O(|t|^n)$ ($n \geq 2$), or of order $O(e^{w|t|})$ ($w > 0$).

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Approximation operators which have been investigated include the Bernstein operators [2], the Baskakov operators [1], the Mirakjan-Szász operators [18, 26] and their generalized forms (see Hsu [12], Schurer [22] and Sikkema [23]), the operators of Meyer-König and Zeller [17, 5, 24, 19, 20], the Gamma operators (see [19]), the Post-Widder operators (see [27, 13]), the Gauss-Weierstrass operators [11, 8],.... All these are special cases of exponential operators which are characterized as integral operators with kernels satisfying a certain type of partial differential equation (cf. May [16] and Ismail and May [13]). They proved that if $\{L_n\}$ is a sequence of exponential operators and if f is a function of order $O(e^{\mu|t|})$ and has a continuous 2nd derivative on the interval $[a, b]$, then $(L_n f)(t)$ converge to $f(t)$ uniformly in any closed subinterval of (a, b).

On the other hand, one can also formulate similar theorems for the approximation of operator-valued functions. For instance, Butzer and Berens [4, pp. 24-29] gave necessary and sufficient conditions for the approximation by operators of the form $(L_n T(\cdot))(t) = \sum_{k=0}^{n} \phi_{n,k}(t)\, T(k/n)$. A special case are the Bernstein operators (with $\phi_{n,k}(t) = \binom{n}{k} t^k(1-t)^{n-k}$); they yield Kendall's representation formula when they are applied to an one-parameter (C_0)-semigroup of operators.

This paper is concerned with generalizations of the known results mentioned to approximations of m-variable real or operator-valued functions, and with the application to representations of operator semigroups. In section 2, we establish two Bohman-Korovkin-type theorems. The first one deals with the approximation problem

$$\lim_{n\to\infty} \int_{\mathbb{R}^m} T(u)x\, d_u m_n(u; t) = T(t)x \qquad (x \in X),$$

where X is a Banach space, $T(t)$ is an unbounded $B(X)$-valued function on $\mathbb{R}^m$, and $\{m_n(.; t); t \in \mathbb{R}^m, n = 1, 2,...,\}$ is a family of positive, finite Borel measures on $\mathbb{R}^m$. The second one will treat the approximation of unbounded m-parameter real functions by a sequence of linear positive operators. In sect. 3 we examine some examples of approximation operators; special cases of them will lead to some of such basic operators as those mentioned above. Finally, in section 4, we apply these approximation operators to derive several representation formulas for strongly continuous m-parameter semigroups of operators; particular cases lead to such exponential formulas for one-parameter semigroups as those of Hille, Phillips, Widder and Kendall (cf. Chung [7]).

2. The Approximation Theorems

Let X be a Banach space and $B(X)$ be the Banach algebra of bounded linear operators on X. $\|\cdot\|$ will be used to denote the norm of X as well as that of $B(X)$. For $t = (t_1, t_2, \ldots, t_m) \in \mathbb{R}^m$, $|t|_p$ $(p \geq 1)$ will denote the norm $(\sum_{i=1}^m |t_i|^p)^{1/p}$; but we simply use $|t|$ for $|t|_2$.

DEFINITION 1. A real function $g(t)$ is said to belong to $G(K)$ (for a set $K \subset \mathbb{R}^m$) if it is positive and strictly convex on $\mathbb{R}^m$, and is continuously differentiable on K, and if in addition, it satisfies the growth condition

$$\lim_{r \to \infty} (\sup\{g(t)/|t| : |t| \geq r\}) = \infty. \tag{1}$$

Notice that $g(\in G(K))$, as a strictly convex function, is continuous on $\mathbb{R}^m$, and that the function

$$h_t(u) = g(u) - [g(t) - \nabla g(t) \cdot (u - t)], \; t \in K, \; u \in \mathbb{R}^m, \tag{2}$$

is positive for $u \neq t$ and is continuous in (t, u) on $K \times \mathbb{R}^m$ (see [10, pp. 12 and 25]).

It is clear that the functions $|t|_p^q$ $(p, q > 1)$ and $e^{w|t|_p}$ $(p > 1, w > 0)$ belong to $G(\mathbb{R}^m)$, and in case $m = 1$, $|t|^q$ $(q > 1)$ and $e^{w|t|}$ $(w > 0)$ belong to $G(\mathbb{R})$. But the functions $|t|_1^q$ and $e^{w|t|_1}$ are not strictly convex if $m > 1$, and $|t|_p$ does not satisfy the condition (1), therefore they are not in $G(K)$ for any K.

Now suppose $\{m_n(\cdot; t) : t \in \mathbb{R}^m, n = 1, 2, \ldots\}$ is a family of positive, finite Borel measures on $\mathbb{R}^m$. Then we define

DEFINITION 2. Given a function $g \in G(K)$, $O_s(g(t), K)$ denotes the set of those $B(X)$-valued functions $T(t)$ defined on $\mathbb{R}^m$ with the properties: (i) for every $x \in X$, the X-valued function $T(t)x$ is strongly measurable with respect to each of the measures $\{m_n(\cdot; t)\}$, (ii) $T(t)$ is bounded on every bounded subset of $\mathbb{R}^m$ and strongly continuous at every point of the set K; (iii) $T(t)$ satisfies

$$\overline{\lim_{r \to \infty}} \{\sup\{\|T(t)\|/g(t) : |t| \geq r\}\} < \infty. \tag{3}$$

The set $O_u(g(t), K)$ is defined similarly with the exceptions that (i) is replaced by the stronger condition that $T(t)$, as a $B(X)$-valued function, is strongly measurable *w. r. t.* $\{m_n(\cdot; t)\}$, and the continuity of $T(t)$ in (ii) is now taken in the sense of operator norm. Note that when $m = 1$, (3) means $T(t) = O(g(t))$ $(|t| \to \infty)$.

If $g(t)$ is integrable with respect to each $m_n(\cdot; t)$, then Def. 2 implies that for each $T(t) \in O_s(g(t), K)$ and for each $x \in X$, the Bochner integral

$$(L_n T(\cdot))(t)x = \int_{\mathbb{R}^m} T(u)x \, dm_n(u; t) \tag{4}$$

exists for every $t \in \mathbb{R}^m$ and $n = 1, 2, \ldots$. Thus (4) defines for every n, a linear operator from $O_s(g(t), K)$ into the set of all $B(X)$-valued functions on $\mathbb{R}^m$. Similar statements apply to $O_u(g(t), K)$.

For simplicity of notation, we will use $(L_n 1)(t)$, $(L_n u_i)(t)$ $(i = 1, 2, \ldots, m)$ and $(L_n g(\cdot))(t)$ to denote the respective Lebesgue integrals of the functions 1, $t_i (i = 1, 2, \ldots, m)$ and $g(t)$ with respect to $m_n(\cdot\,; t)$.

THEOREM 2.1. *Let the function $g \in G(K)$ for a compact set K of $\mathbb{R}^m$. Suppose that g is integrable with respect to each member of the family $\{m_n(\cdot\,; t)$: $t \in \mathbb{R}^m, n = 1, 2, \ldots\}$ of positive, finite Borel measures so that a sequence $\{L_n\}$ of operators can be defined by* (4) *on $O_s(g(t), K)$. Then the following statements are equivalent:*

(i) *For any $T(t) \in O_s(g(t), K)$ and any $x \in X$,*

$$\lim_{n \to \infty} (L_n T(\cdot))(t)x = T(t)x \qquad \text{uniformly for } t \text{ in } K. \tag{5}$$

(ii) *The limits in* (6), (7) *and* (8) *hold uniformly for t in K.*

$$\lim_{n \to \infty} (L_n 1)(t) = 1; \tag{6}$$

$$\lim_{n \to \infty} (L_n u_i)(t) = t_i, \qquad i = 1, 2, \ldots, m; \tag{7}$$

$$\lim_{n \to \infty} (L_n g(\cdot))(t) = g(t). \tag{8}$$

(iii) *$(L_n 1)(t) \to 1$ and $(L_n h_t(\cdot))(t) \to 0$ uniformly for t in K as $n \to \infty$, where $h_t(u)$ is defined in* (2).

Moreover, the theorem remains valid when $O_s(g(t), K)$ is replaced by $O_u(g(t), K)$ and the limit in (5) *is replaced by one taken in the sense of uniform operator topology.*

We first prove the following

LEMMA. *Let K, g, and $h_t(\cdot)$ be as assumed in Theorem 2.1. Then, for any $T(t) \in O_s(g(t), K)$, $x \in X$ and any $\delta > 0$, there exists a constant $M(T(\cdot)x, K, \delta)$ such that*

$$\sup_{|u-t| \geq \delta} \frac{\| T(u)x - T(t)x \|}{h_t(u)} \leq M(T(\cdot)x, K, \delta) \qquad (u \in \mathbb{R}^m, t \in K). \tag{9}$$

Proof. Since $T(t)x$, $g(t)$ and $\nabla g(t)$ are continuous on the compact set K, they are bounded there by a sufficiently large number C. Hence we have the following estimate

$$\frac{\| T(u)x - T(t)x \|}{h_t(u)} \leq \frac{\| T(u)x \| + C}{g(u)} \left| 1 - \frac{C}{g(u)} - C\frac{| u | - C}{g(u)} \right|^{-1}$$

for all $t \in K$ and all large $u \in \mathbb{R}^m$. Now the assumptions (1), (3) imply the existence of a positive number $M_1(K)$ such that $\overline{\lim}_{r\to\infty}\{\sup\{\|T(u)x - T(t)x\|/h_t(u) : |u| \geq r, t \in K\}\} \leq M_1(K)$. Therefore, there is a $r_0 > 0$ such that $\|T(u)x - T(t)x\|/h_t(u) < M_1(K) + 1$ for all $t \in K$ and for all u outside the sphere $\{u \in \mathbb{R}^m; |u| \leq r_0\}$. As noted before, $h_t(u)$ is a positive, continuous function on the compact set $\{(u, t) : |u| \leq r_0, t \in K$ and $|u - t| \geq \delta\}$, hence $h_t(u)$ assumes a positive minimum m_1. It is now easy to see that one can take $M_1 + 1 + 2\sup\{\|T(t)x\| : t \in K$ or $|t| \leq r_0\}/m_1$ as the required number $M(T(\cdot)x, K, \delta)$.

Proof of Theorem 2.1. First, suppose (i) is true. Let $x \in X$ and $x^* \in X^*$ be such that $x^*(x) = 1$. If we apply x^* to both sides of (5) while $T(t)$ is substituted by I, or $t_i I$, or $g(t) I$, where I is the identity operator of $B(X)$, then we obtain (6), (7) and (8). Hence (i) implies (ii). (ii) $\Rightarrow$ (iii) being obvious, it remains to verify the part (iii) $\Rightarrow$ (i). For $t \in K$, $x \in X$, we have

$$
\begin{aligned}
\|(L_nT(\cdot))(t)x - T(t)x\| &\leq \left\|\int [T(u)x - T(t)x]\, d_u m_n(u; t)\right\| \\
&\quad + |(L_n1)(t) - 1|\, \|T(t)x\| \\
&\leq J_1 + J_2 + |(L_n1)(t) - 1|\, \|T(t)x\|,
\end{aligned}
$$

where

$$
\begin{aligned}
J_1 &= \int_{|u-t| \leq \delta} \|T(u)x - T(t)x\|\, d_u m_n(u; t) \\
&\leq \omega(\delta, T(\cdot)x, K) \cdot (L_n1)(t), \\
J_2 &= \int_{|u-t| > \delta} \|T(u)x - T(t)x\|\, d_u m_n(u; t) \\
&\leq \sup_{|u-t| > \delta} \frac{\|T(u)x - T(t)x\|}{h_t(u)} (L_n h_t(\cdot))(t) \\
&\leq M(T(\cdot)x, K, \delta)(L_n h_t(\cdot))(t).
\end{aligned}
$$

Here $\omega(\delta, T(\cdot)x, K)$ represents the modulus of continuity of $T(t)x$ defined as $\sup\{\|T(u)x - T(t)x\| ; u \in \mathbb{R}^m,\ t \in K$ and $|u - t| \leq \delta\}$: it tends to 0 with δ because $T(u)x$ is continuous at every point of the compact set K. This fact together with the above estimates and the assumption that $(L_n1)(t) \to 1$ and $(L_n h_t(\cdot))(t) \to 0$ uniformly for $t \in K$ implies that $(L_nT(\cdot))(t)x$ converges to $T(t)x$ uniformly for $t \in K$.

Before stating our second theorem, we describe first those functions which will be involved.

DEFINITION 3. Let $g \in G(K)$. By $O(g(t), K)$ we mean the set of those real-valued functions $f(t)$ defined on $\mathbb{R}^m$ such that $f(t)$ is bounded on every bounded subset of $\mathbb{R}^m$, continuous at every point of K, and satisfying the condition (3) with $\|T(t)\|$ there replaced by $|f(t)|$.

THEOREM 2.2. *Let $g(t)$ be in $G(K)$ for a compact set K, $\{L_n\}$ be a sequence of linear positive operators, $L_n : O(g(t), K) \to C(K)$. Then $L_n f(t)$ converges to $f(t)$ uniformly on K for any $f \in O(g(t), K)$ if and only if it is true for the functions* 1, t_i $(i = 1, 2, \ldots, m)$ *and* $g(t)$, *or, if and only if* $(L_n 1)(t) \to 1$ *and* $(L_n h_t(\cdot))(t) \to 0$ *uniformly on* K.

The estimate in (9) plays an important role in the proof of Theorem 2.1, it is also the same estimate which enables us to implement the proof of Theorem 2.2. By the same proof as in the previous lemma, we can find for given $\delta > 0$, compact set K and $f \in O(g(t), K)$ a constant $M(f, K, \delta)$ such that

$$|f(u) - f(t)| \leqslant M(f, K, \delta)\, h_t(u) \tag{10}$$

holds for all $t \in K$, $u \in \mathbb{R}^m$ with $|u - t| > \delta$.

Proof of Theorem 2.2. While the other two implications are obvious, we will only prove the crucial part that $(L_n 1)(t) \to 1$ and $(L_n h_t(\cdot))(t) \to 0$ uniformly on K implies for any $f \in O(g(t), K)$ $(L_n f(\cdot))(t) \to f(t)$ uniformly on K. Let $f \in O(g(t), K)$. Then it follows from (10) that for all $t \in K$ and $u \in \mathbb{R}^m$ we have

$$|f(u) - f(t)| \leqslant \omega(\delta, f(\cdot), K) + M(f, K, \delta)\, h_t(u),$$

thus

$$-\omega(\delta) - M(f, K, \delta)\, h_t(u) \leqslant f(u) - f(t) \leqslant \omega(\delta) + M(f, K, \delta)\, h_t(u), \tag{11}$$

where $\omega(\delta) = \omega(\delta, f(\cdot), K)$ is the modulus of continuity of $f(t)$. On applying L_n to (11) we have

$$|(L_n f(\cdot))(t) - f(t)(L_n 1)(t)| \leqslant \omega(\delta)(L_n 1)(t) + M(f, K, \delta)(L_n h_t(\cdot))(t).$$

It follows that

$$\begin{aligned}|(L_n f(\cdot))(t) - f(t)| &\leqslant \omega(\delta)(L_n 1)(t) + |f(t)|\,|(L_n 1)(t) - 1| \\ &\quad + M(f, K, \delta)(L_n h_t(\cdot))(t)\end{aligned} \tag{12}$$

which, like the situation in the proof of Theorem 2.1, implies the uniform convergence of $(L_n f(\cdot))(t)$ to $f(t)$ on K. Hence the theorem is proved.

Remark. If for a certain value of n, $(L_n f(\cdot))(t) = f(t)$ $(t \in K)$ for $f(u) = 1$, u_i $(i = 1, 2, \ldots, m)$, $g(u)$, then we have from (2) that $(L_n h_t(\cdot))(t) = 0$; this together with (12) implies that for this n we have the equality $(L_n f(\cdot))(t) = f(t)$ $(t \in K)$ for every $f \in O(g(t), K)$. Similar assertions apply to the operators in Theorem 2.1.

3. Some Approximation Operators

Whether a sequence $\{L_n\}$ of operators will approximate functions in $O(g(t), K)$ or $O_s(g(t), K)$ or $O_u(g(t), K)$ has been shown to be determined by whether these operators satisfy (6), (7) and (8). In this section we will give some examples of such operators. For simplicity of exposition and for applications in section 4, we will mention in our theorems only those assertions for functions in $O_s(g(t), K)$: similar assertions concerning $O_u(g(t), K)$ or $O(g(t), K)$ may surely be formulated without difficulties by the reader.

The following notations are used in the rest of this paper. For $t \in \mathbb{R}^m$, $\bar{t}$ denotes the number $\sum_{i=1}^{m} t_i$ and $\mathbb{R}_t^m$ denotes the set $\{u \in \mathbb{R}^m; u_i \geq t_i, i = 1, 2, \dots, m\}$. If $k = (k_1, k_2, \dots, k_m)$, k_i's being nonnegative integers, and n is any integer, then $\binom{n}{k}$ will represent the number

$$\frac{n(n-1)\cdots(n-\bar{k}-1)}{k_1!\,k_2!\cdots k_m!}.$$

(I) *The polynomial distributions.* Let $\{\lambda_n\}$ be a sequence of positive numbers, and $K(\lambda_n) = \{t \in \mathbb{R}_a^m : \overline{t-a} \leq \lambda_n\}$. If $t \in K(\lambda_n)$, then $m_n^1(\cdot; t)$ is a discrete measure:

$$m_n^1(u; t) = \begin{cases} \Phi_{n,k}^1(t) = \binom{n}{k} \prod_{i=1}^{m} \left(\dfrac{t_i - a_i}{\lambda_n}\right)^{k_i} \left(1 - \dfrac{\overline{t-a}}{\lambda_n}\right)^{n-\bar{k}} & \text{for } u = a + \dfrac{\lambda_n}{n} k, \ \bar{k} \leq n, \\ 0 & \text{for } u \text{ elsewhere.} \end{cases}$$

If $t \notin K(\lambda_n)$, $m_n^1(\cdot; t)$ is a zero measure.

The associated operators are

$$(L_n^1 T(\cdot))(t)x = \begin{cases} \sum_{\bar{k} \leq n} T(a + k\lambda_n/n)\, x\Phi_{n,k}^1(t) & \text{for } t \in K(\lambda_n); \\ 0 & \text{for } t \notin K(\lambda_n). \end{cases}$$

These operators reduce to the original Bernstein operators in case $m = 1$, $a = 0$ and $\lambda_n = 1$ ($n = 1, 2, \dots$). In the following Theorems 3.1 and 3.2 we will give conditions on λ_n such that the limit

$$\lim_{n \to \infty} (L_n^1 T(\cdot))(t)x = T(t)x \tag{13}$$

holds uniformly on compact sets for various classes of functions.

THEOREM 3.1. *If $\lambda_n/n \to 0$ as $n \to \infty$, and if K is any compact set which is contained in $K(\lambda_n)$ for all large n, then for any $T(t) \in O_s(|t|^r, K)$ and any $x \in X$, the limit* (13) *holds uniformly on K.*

Proof. Notice first that: a) $O_s(\| t \|_j^p, K)$ is independent of $j (\geq 1)$ since all kinds of norms in $\mathbb{R}^m$ are equivalent; b) if $p \leqslant q$, then $O_s(\| t \|^p) \subset O_s(\| t \|^q)$; c) $\| t \|^p, p > 1$, is continuously differentiable on $\mathbb{R}^m$. Thus, in view of Therem 2.1, it suffices to show that (6), (7) and (8) with $g_p(t) = \| t \|_p^p = \sum_{i=1}^m |t_i|^p$ converge uniformly on any compact K which is contained in $K(x_n)$ for all large n. But this is equivalent (because of Theorem 2.1) to showing that for each $p = 0, 1, 2, \ldots$ and each $i = 1, 2, \ldots, m$,

$$\lim_{n \to \infty} (L_n^1 u_i^p)(t) = t_i^p \tag{14}$$

will converge uniformly for t in K. Due to symmetry, proving (10) for the case $i = 1$ suffices.

Without loss of generality, we may assume $a = 0$ (otherwise, a translation of variable will make it so). Let $p = 0$. We have $(L_n^1 1)(t) = \sum_{\bar{k} \leqslant n} \Phi_{n,k}^1(t) = (\bar{t}/x_n - 1 - \bar{t}/x_n)^n = 1$ for $\bar{t} \leqslant x_n$, $= 0$ otherwise. To prove (14) by induction, we assume it to be true for all $p \leqslant j - 1$. Since $y^j = y(y-1)(y-2) \cdots (y - j + 1) + \sum_{i=1}^{j-1} c_i y^i$ for some constants c_i, there follows:

$$\begin{aligned}
(L_n^1 u_1^j)(t) &= \sum_{\bar{k} \leqslant n} \left(\frac{x_n k_1}{n}\right)^j \Phi_{n,k}^1(t) \\
&= (x_n/n)^j \sum \left\{ k_1(k_1 - 1) \cdots (k_1 - j - 1) + \sum_{i=1}^{j-1} c_i k_1^i \right\} \Phi_{n,k}^1(t) \\
&= J_1 + J_2,
\end{aligned}$$

where

$$\begin{aligned}
J_1 &= \frac{n \cdots (n - j - 1)}{n^j} \sum_{k_1 \geqslant j} x_n^j \frac{k_1 \cdots (k_1 - j + 1)}{n \cdots (n - j + 1)} \binom{n}{k} \prod_{i=1}^m \left(\frac{t_i}{x_n}\right)^{k_i} \left(1 - \frac{\bar{t}}{x_n}\right)^{n - \bar{k}} \\
&= \frac{n \cdots (n - j - 1)}{n^j} t_1^j \cdot \sum_{\bar{k} \leqslant n - j} \Phi_{n-j,k}^1(t) \to t_1^j.
\end{aligned}$$

$$\begin{aligned}
J_2 &= \sum_{i=1}^{j-1} c_i \left(\frac{x_n}{n}\right)^{j-i} \sum \left(\frac{x_n k_1}{n}\right)^i \Phi_{n,k}^1(t) = \sum_{i=1}^{j-1} c_i \left(\frac{x_n}{n}\right)^{j-i} (L_n^1 u_1^i)(t) \\
&\to \sum c_i \cdot 0 \cdot t_1^i = 0 \qquad \text{as} \quad n \to \infty.
\end{aligned}$$

Hence the uniform convergence (on compact subsets of $\mathbb{R}_{a+}^m$) of the limit (14) is true for $p = j$, and it is true inductively for all $p = 1, 2, \ldots$. This proves the theorem.

COROLLARY 3.2. *If $x_n = x$, $n = 1, 2, \ldots$, then for every $T(t)$ which is strongly continuous on $K(x)$, the limit* (13) *holds uniformly on* $K(x)$.

COROLLARY 3.3. *If $x_n/n \to 0$ and $x_n \to \infty$ as $n \to \infty$, and if $T(t)$ is a* *strongly continuous function on* $\mathbb{R}^m_{a+}$ *such that* $\|T(t)\| = O(|t|^p)$ $(|t| \to \infty)$, $p \geq 1$, *then* (13) *holds uniformly on any compact subset of* $\mathbb{R}^m_{a+}$.

The next theorem shows in particular that in case $m = 1$ Corollary 3.3 holds also for those continuous functions with order $O(e^{w \cdot t})$.

THEOREM 3.4. *Let* $m = 1$. *If* $\alpha_n/n \to 0$ *and* $\alpha_n \to \infty$ *as* $n \to \infty$, *then for any compact set* $K \subset \mathbb{R}^m_{a-}$ *and any* $T(t) \in O_s(e^{w \cdot t}, K)$, *the limit* (13) *holds uniformly for* t *in* K.

Proof. Since $e^{w \cdot t} \in G(R)$, the theorem will follow from Theorem 2.1 once we prove

$$\lim_{n\to\infty} \sum_{k=0}^{n} \exp\left(w\alpha_n \frac{k}{n}\right)\binom{n}{k}\left(\frac{t}{\alpha_n}\right)^k\left(1 - \frac{t}{\alpha_n}\right)^{n-k} = e^{wt} \tag{15}$$

uniformly on $[0, b]$ for any $b > 0$. Here we have assumed $a = 0$ without loss of generality. The summation in (15) is equal to

$$\begin{aligned}
&\left[\frac{t}{\alpha_n}\exp(w\alpha_n/n) + 1 - \frac{t}{\alpha_n}\right]^n \\
&\quad = \exp\left\{\alpha_n \cdot \ln\left[1 + \frac{t}{\alpha_n}(\exp(w\alpha_n/n) - 1)\right] \Big/ \left(\frac{\alpha_n}{n}\right)\right\} \\
&\quad = \exp\{wte^{w\lambda}/[1 + t(e^{w\lambda} - 1)/\alpha_n]\}
\end{aligned}$$

for some $0 < \lambda = \lambda_n < \alpha_n/n$, the mean value theorem being used. From this and the estimate: $|e^{w\lambda}/[1 - t(e^{w\lambda} - 1)/\alpha_n] - 1| \leq (1 - (t/\alpha_n))(e^{w\lambda} - 1)$, $t \geq 0$, together with the hypothesis on α_n (15) follows immediately.

Remark. When $m > 1$, it is also possible to verify that $\lim_{n\to\infty} (L_n^1 e^{w \cdot 1})(t) = e^{w \cdot t}$ converges uniformly on compact subsets of $\mathbb{R}^m_{a-}$. However, we could not assert from it that a proposition similar to Theorem 3.4 be true because Theorem 2.1 does not apply to $e^{w \cdot t}$ which is not strictly convex as required in that theorem.

(II) *The negative polynomial distributions.* Let $\{\alpha_n\}$ be a sequence of positive numbers. $\{m_n^2(\cdot\,; t)\}$ is defined as follows:

If $t \in \mathbb{R}^m_{a-}$, then the measure $m_n^2(\cdot\,; t)$ is defined as

$$m_n^2(u; t) = \begin{cases} \Phi^2_{n,k}(t) = \binom{-n}{k}\prod_{i=1}^{m}\left(\dfrac{a_i - t_i}{\alpha_n}\right)^{k_i}\left(1 - \dfrac{\overline{t - a}}{\alpha_n}\right)^{-n-\bar{k}} & \\ & \text{for } u = a - \dfrac{\alpha_n}{n}k,\ \bar{k} \geq 0 \\ 0 & \text{for } u \text{ elsewhere.} \end{cases}$$

If $t \notin \mathbb{R}^m_{a-}$, then we define $m_n^2(\cdot\,; t)$ to be 0 identically.

Thus, we have defined such linear operators as

$$(L_n{}^2T(\cdot))(t)x = \begin{cases} \sum_{k\geqslant 0} T(a + k\,\alpha_n/n)\, x\Phi^2_{n.k}(t) & \text{for} \quad t \in \mathbb{R}^m_{a+}\,; \\ 0 & \text{for} \quad t \notin \mathbb{R}^m_{a+}\,; \end{cases}$$

these lead to the special Baskakov operators in case $m = 1$ and $\alpha_n = 1$, $n = 1, 2, \ldots$, (recall [1]).

THEOREM 3.5. *If $\lim_{n\to\infty} \alpha_n/n = 0$, $\underline{\lim}\, \alpha_n > 0$, and if K is a compact set of $\mathbb{R}^m_{a+}$, then for any $T(t) \in O_s(|t|^p, K)(p > 1)$ and any $x \in X$,*

$$\lim_{n\to\infty} (L_n{}^2T(\cdot))(t)x = T(t)x \tag{16}$$

uniformly for t in K.

Proof. The proof is quite the same as that of Theorem 3.1. First, denoting $(t - a)/\alpha_n$ by b, we have for all $t \in \mathbb{R}^m_{a+}$

$$\begin{aligned}(L_n{}^2 1)(t) &= \sum_{\bar{k}\geqslant 0} \binom{-n}{k} \prod_{i=1}^{m} (-b_i)^{k_i}(1 - \bar{b})^{-n-\bar{k}} \\ &= \sum_{\nu=0}^{\infty} \binom{-n}{\nu}\left[\sum_{\bar{k}=\nu} \binom{\nu}{k} \prod_{i=1}^{m} (-b_i)^{k_i}\right](1 + \bar{b})^{-n-\nu} \\ &= \sum_{\nu=0}^{\infty} \binom{-n}{\nu} (-\bar{b})^{\nu}(1 - \bar{b})^{-n-\nu} = (-\bar{b} - 1 + \bar{b})^{-n} = 1.\end{aligned}$$

Next, by a similar computation as in the proof of Theorem 3.1, we get

$$(L_n{}^2u_1{}^j)(t) = \frac{n \cdots (n + j - 1)}{n^j}\, t_1{}^j(L^2_{n+j}1)(t) - \sum_{i=1}^{j-1} c_i \left(\frac{x_n}{n}\right)^{j-i} (L_n{}^2u_1{}^i)(t).$$

This converges to $t_i{}^j$ uniformly on compact sets, by the induction assumption. Hence, the theorem follows (see the proof of Theorem 3.1).

THEOREM 3.6. *In case $m = 1$, the class $O_s(|t|^p, K)$ in Theorem 3.5 may be replaced by the larger class $O_s(e^{w|t|}, K)(w > 0)$.*

Proof. In view of Theorems 2.1 and 3.5 we only have to establish the uniform convergence of $(L_n{}^2e^{wu})(t)$ to e^{wt} on $[0, b]$ for any $b > 0$. By a computation like that in the proof of the last theorem and then by the mean value theorem we have for each t and each n a corresponding λ between 0 and α_n/n such that

$$(L_n^2 e^{wu})(t) = \left[1 + \frac{t}{\alpha_n} - \frac{t}{\alpha_n}\exp(w\alpha_n/n)\right]^{-n}$$
$$= \exp\{wte^{w\lambda}/[1 - t(1 - e^{w\lambda})/\alpha_n]\}.$$

The assertion then follows from this and the estimate

$$|(e^{w\lambda}/[1 - t(1 - e^{w\lambda})/\alpha_n]) - 1| = |(1 + t/\alpha_n)(e^{w\lambda} - 1)|/|1 - t(e^{w\lambda} - 1)/\alpha_n|;$$

since the right term tends to 0 uniformly on $[0, b]$ as $n \to \infty$.

(III) *The Poisson distributions.* These are the measures $m_n^3(\cdot; t)$ defined as follows: If $t \in \mathbb{R}^m_{a+}$, then

$$m_n^3(u; t) = \begin{cases} \Phi^3_{n,k}(t) = e^{n(\overline{a-t})} \prod_{i=1}^{m} (n(t_i - a_i))^{k_i}/k_i\,! & \text{for } u = a + \dfrac{k}{n},\ \bar{k} \geq 0, \\ 0 & \text{for } u \text{ elsewhere;} \end{cases}$$

if $t \notin \mathbb{R}^m_{a+}$, $m_n^3(\cdot; t)$ is defined to be 0 identically.

The corresponding linear operators are

$$(L_n^3 T(\cdot))(t)x = \begin{cases} \sum_{k \geq 0} T\left(a + \dfrac{k}{n}\right) x\Phi^3_{n,k}(t) & \text{for} \quad t \in \mathbb{R}^m_{a+}; \\ 0 & \text{for} \quad t \notin \mathbb{R}^m_{a+}. \end{cases} \tag{17}$$

In case $m = 1$, $a = 0$ they become the Mirakjan-Szász operators ([18], [26]). Now, following the same way of proof as that in Theorem 3.1, we have

THEOREM 3.7. *If $T(t) \in O_s(|t|^p, K)$ $(p \geq 1)$ where K is a compact subset of $\mathbb{R}^m_{a+}$, then $(L_n^3 T(\cdot))(t)\,x$ converges uniformly to $T(t)\,x$ on K.*

THEOREM 3.8. *In case $m = 1$, the class $O_s(|t|^p, K)$ in Theorem 3.7 may be replaced by the class $O_s(e^{w|t|}, K)$.*

Proof. In view of Theorem 2.1 and 3.7, this follows from the fact that $(L_n^3 e^{w|u|})(t)$ converges to $e^{w|t|}$ uniformly for t in K (cf. [16]).

(IV) *The negative polynomial distributions of the second kind.* We define $\{m_n^4(\cdot; t)\}$ as follows: if $t \in K(1) = \{t \in \mathbb{R}^m_{a+} : \overline{t - a} \leq 1\}$,

$$m_n^4(u;t) = \begin{cases} \Phi^4_{n,k}(t) = (1-\overline{t-a})^{n+1}\dbinom{-n-1}{k}\displaystyle\prod_{i=1}^m (a_i - t_i)^{k_i} & \\ & \text{for } u = a - \dfrac{k}{n+\bar k},\ \bar k \geqslant 0,\\ 0 & \text{for other } u.\end{cases}$$

If $t \notin K(1)$, then $m_n^4(\cdot;t)$ is defined to be 0 identically. The corresponding operators are

$$(L_n^4T(\cdot))(t)x = \begin{cases} \sum_{\bar k\geqslant 0} T(a + k/(n-\bar k))\,\Phi^4_{n,k}(t)x & \text{for } t \in K(1),\\ 0 & \text{for } t \notin K(1)\end{cases} \tag{18}$$

which map the set of strongly continuous functions on $K(1)$ into itself. When $m = 1$, these are the Meyer-König and Zeller operators (recall Sec. 1).

THEOREM 3.9. *If $T(t)$ is strongly continuous on the set $K(1)$, then for any $x \in X$, $(L_n^4T(\cdot))(t)\,x$ converges to $T(t)\,x$ uniformly on $K(1)$.*

Proof. Since $m_n^4(u;t) = 0$ for all u outside $K(1)$, the behavior of $T(u)$ for $u \notin K(1)$ is irrelavent to $L_n^4T(\cdot)$, and so we may assume that $T(t)$ is bounded on $\mathbb{R}^m$. Thus, by Theorem 2.1 and the remark in the proof of Theorem 3.1, we need only verify that $(L_n^4u_1^i)(t)$ converges uniformly on $K(1)$ to t_1^i for $i = 0$, 1 and 2. First,

$$\begin{aligned}(L_n^4 1)(t) &= (1-\overline{t-a})^{n+1}\sum_{\bar k\geqslant 0}\binom{-n-1}{k}\prod_{i=1}^m (a_i - t_i)^{k_i}\\ &= (1-\overline{t-a})^{n+1}(1+\overline{a-t})^{-n-1} = 1.\end{aligned}$$

To prove the assertion for $i = 1$ and 2, we will assume for simplicity of computations that $a = 0$; there will be no loss of generality. Indeed,

$$\begin{aligned}(L_n^4u_1)(t) &= (1-\bar t)^{n+1}\sum_{\bar k\geqslant 0}\binom{-n-1}{k}\frac{k_1}{n+\bar k}\prod_{i=1}^m (-t_i)^{k_i}\\ &= t_1(1-\bar t)^{n+1}\sum_{\substack{\bar k\geqslant 1\\ k_1\geqslant 1}}\frac{(n+1)\cdots(n+\bar k-1)}{(k_1-1)!\,k_2!\cdots k_m!}\,t_1^{k_1-1}\prod_{i\neq 1} t_i^{k_i} = t_1\,.\end{aligned}$$

Similarly we have $\sum_k [k_1(k_1-1)/(n-\bar k)(n+\bar k-1)]\,\Phi^4_{n,k}(t) = t_1^2$. But since the absolute value of the difference between $(k_1/(n+\bar k))^2$ and $k_1(k_1-1)/(n+\bar k)(n+\bar k-1)$ is no larger than $1/n$ for any k, we have $|(L_n^4u_1^2)(t) - t_1^2| \leqslant 1/n\sum \Phi^4_{n,k}(t) = 1/n$; this shows that $(L_n^4u_1^2)(t)$ converges uniformly to t_1^2.

(V) *The Gamma distribtions I.* Let $\{m_n{}^1(\cdot;t)\}$ be so defined that the corresponding operators are given by

$$(L_n{}^5T(\cdot))(t)x = \int_0^\infty \cdots \int_0^\infty \exp\left[-\sum_{i=1}^m (nu_i/t_i)\right] \prod_{i=1}^m [(nu_i/t_i)^n/u_i(n-1)!]$$

$$\cdot\, T(u)x\, du_1 \cdots du_m\,, \qquad t_i \in (0, \infty), \qquad i = 1, 2, \ldots, m. \quad (19)$$

It is not hard to see that when $T(t)$ satisfies (3) with $g(t) = e^{w|t|}$, the above integral exists for large n. Notice also that in case $m = 1$, the integral becomes

$$(L_n{}^5T(\cdot))(t)x = \frac{(n/t)^n}{(n-1)!} \int_0^\infty e^{-nu/t} u^{n-1} T(u)x\, du, \qquad t > 0, \qquad (20)$$

This is the Post-Widder operator.

THEOREM 3.10. *Let $m = 1$ and K be a compact subset of $(0, \infty)$. Then $(L_n{}^5T(\cdot))(t)x$ converges to $T(t)x$ uniformly on K, provided $T(t)$ belongs to $O_s(e^{w|t|}, K)$.*

Proof. After an $(n-1)$-fold differentiation of two sides of the identity $\int_0^\infty e^{-su}e^{wu}\,du = (s-w)^{-1}$ $(s > w)$, we obtain

$$\int_0^\infty e^{-su}u^{n-1}e^{wu}\,du = (n-1)!\,(s-w)^{-n}, \qquad (21)$$

On substituting $s = n/t$ into (21), we derive that $(L_n{}^1e^{w\cdot u})(t) = (1 - wt/n)^{-n}$. Similarly, $(L_n{}^1ue^{w\cdot u})(t) = t(1 - wt/n)^{-n-1}$. It follows that $(L_n{}^11)(t) = 1$, $(L_n{}^5u)(t) = t$ and that $(L_n{}^1e^{w|u|})(t)$ converges to $e^{w\,t}$ uniformly on K. The theorem now follows from Theorem 2.1.

THEOREM 3.11. *Let $m > 1$ and K be a compact subset of $\prod_{i=1}^m (0, \infty)$. Then $(L_n{}^5T(\cdot))(t)x$ converges to $T(t)x$ uniformly on K, provided $T(t)$ belongs to $O_s(\|t\|^p, K)(p > 1)$.*

Proof. We have for every $j \geq 0$

$$(L_n{}^5u_1{}^j)(t) = \frac{(n/t_1)^n}{(n-1)!} \int_0^\infty e^{-nu_1/t_1} u_1^{n-1} u_1{}^j\, du_1\,,$$

the right-hand term converging uniformly on K to $t_1{}^j$ (by Theorems 2.1 and 3.10). Hence Theorem 2.1 applies again to yield the assertion (recall the remark in the proof of Theorem 3.1).

(VI) *The Gamma distributions II.* Next, we consider the operators defined by

$$(L_n^6T(\cdot))(t)x = \int_0^\infty \cdots \int_0^\infty \exp\left[-\sum_{i=1}^m (nt_i/u_i)\right] \prod_{i=1}^m [(nt_i/u_i)^n/u_i(n-1)!]$$
$$\cdot\, T(u)x\, du_1 \cdots du_m\,. \qquad t_i \in [0, \infty), \qquad i = 1, 2,\ldots, m. \quad (22)$$

One can see easily that $L_n^6T(\cdot)$ is well-defined for $T(t) \in O_s(\| t \|^p, K)$ for all large n. By changes of variables such as $u_i = n/v_i$, $i = 1, 2,\ldots, m$, we obtain a perhaps more convenient expression for L_n^6, namely,

$$(L_n^6T(\cdot))(t)x = \int_0^\infty \cdots \int_0^\infty \exp\left(-\prod_{i=1}^m t_iv_i\right) \prod_{i=1}^m [t_i^n v_i^{n-1}/(n-1)!]$$
$$\cdot\, T\left(\frac{n}{v_1},\ldots, \frac{n}{v_m}\right) x\, dv_1 \cdots dv_m\,. \qquad (23)$$

When $m = 1$, (23) becomes the Gamma operator

$$(L_n^6T(\cdot))(t)x = \frac{t^n}{(n-1)!} \int_0^\infty e^{-tv}v^{n-1}T\left(\frac{n}{v}\right) x\, dv, \qquad t \geqslant 0. \qquad (24)$$

THEOREM 3.12. *Let K be a compact subset of $\mathbb{R}_{0+}^m$. Then for any $T(t) \in O_s(\| t \|^p, K)$ and for any $x \in X$, $(L_n^6T(\cdot))(t)\, x$ converges to $T(t)\, x$ uniformly on K.*

Proof. From (21) we have $(t_i^n/(n-1)!) \int_0^\infty e^{-t_iv_i}v_i^{n-1}dv_i = 1$ for all $n = 1, 2,\ldots$. (Or one can prove it by using integration by parts.)

$$\begin{aligned}(L_n^6v_1^j)(t) &= (t_1^n/(n-1)!) \int_0^\infty \exp(-t_1v_1)\, v_1^{n-1}(n/v_1)^j\, dv_1\\
&= \frac{n^j}{(n-1)\cdots(n-j)}\, t_1^{\,j} \cdot (t_1^{n-j}/(n-j-1)!) \int_0^\infty e^{-t_1v_1}v_1^{n-j-1}\, dv_1\\
&= t_1^{\,j}\, n^j/(n-1)\cdots(n-j) \to t_1^{\,j}, \qquad j = 0, 1,\ldots,\end{aligned}$$

as $n \to \infty$, the convergence of the limit being uniform for t in K. The theorem now follows from Theorem 2.1.

(VII) *The normal distributions.* Let

$$D_um_n^7(u; t) = \left(\frac{n}{2\pi}\right)^{m/2} \exp\left(-\frac{n\, \| u-t \|^2}{2}\right) \qquad (u, t \in \mathbb{R}^m).$$

Then the corresponding operators are defined as

$$(L_n^7T(\cdot))(t)x = \left(\frac{n}{2\pi}\right)^{m/2} \int_{R^m} \exp(-n\, \| u-t \|^2/2)\, T(u)x\, du, \qquad t \in \mathbb{R}^m; \qquad (25)$$

when $m = 1$, these are the Gauss-Weierstrass operators. (see [13].)

THEOREM 3.13. *Let K be a compact set in $\mathbb{R}^m$, and $T(t) \in O_s(\exp(w|t|^2), K)$. Then for any $x \in X$, $(L_n^7 T(\cdot))(t)\,x$ converges to $T(t)\,x$ as $n \to \infty$ uniformly on K.*

Proof. Due to Theorem 2.1, it suffices to show the uniform convergence of $(L_n^7 f(\cdot))(t)$ to $f(t)$ on K for $f(t) = 1, t_i$ $(i = 1,\ldots, m)$ and $\exp(w|t|^2)$ $(w \geq 0)$. One has

$$(L_n^7 1)(t) = \prod_{i=1}^{m} (n/2\pi)^{1/2} \int_{-\infty}^{\infty} \exp[-n(u_i - t_i)^2/2]\, du_i = 1.$$

Since $(L_n^7 u_i)(t)$ is just the mean of the one-dimensional normal distribution $N(t_i, 1/n)$, it is the function t_i. Finally, if we put $\beta_n = 1 - 2w/n$, then

$$\begin{aligned}
&(L_n^7 \exp(w|t|^2))(t)\\
&\quad = \prod_{i=1}^{m} (n/2\pi)^{1/2} \int_{-\infty}^{\infty} \exp[-n(u_i - t_i)^2/2] \exp(w u_i^2)\, du_i\\
&\quad = \prod_{i=1}^{m} (n/2\pi)^{1/2} \int_{-\infty}^{\infty} \exp\left(-\frac{n}{2}\beta_n(u_i - t_i/\beta_n)^2\right) \exp(w t_i^2/\beta_n)\, du_i\\
&\quad = \prod_{i=1}^{m} (\beta_n)^{-1/2} \exp(w t_i^2/\beta_n) = (\beta_n)^{-(1/2)m} \exp(w|t|^2/\beta_n).
\end{aligned}$$

Since β_n tends to 1 when $n \to \infty$, the last term converges to $\exp(w|t|^2)$ uniformly on K.

(VIII). In this example we will consider operators $\{L_n^8\}$ which are defined for $t \geq 0$ by

$$\begin{aligned}
(L_n^8 T(\cdot))(t)x &= \int_0^{\infty} e^{-n(t+u)} \sum_{k=0}^{\infty} n^{2k} \frac{t^k u^{k-1}}{k!\,(k-1)!} T(u)x\, du\\
&= e^{-nt} \sum_{k=0}^{\infty} \frac{(nt)^k}{k!} \frac{n^k}{(k-1)!} \int_0^{\infty} e^{-nu} u^{k-1} T(u)x\, du. \qquad (26)
\end{aligned}$$

THEOREM 3.14. *Let K be any compact subset of $[0, \infty)$, $T(t) \in (O_s(e^{w|t|}), K)$. Then for $x \in X$, $(L_n^8 T(\cdot))(t)\,x$ converges to $T(t)\,x$ uniformly on K.*

Proof. Using (21), we have

$$\begin{aligned}
(L_n^8 e^{wu})(t) &= e^{-nt} \sum_{k=0}^{\infty} \frac{(n^2 t)^k}{k!} \frac{1}{(k-1)!} \int_0^{\infty} e^{-nu} u^{k-1} e^{wu}\, du\\
&= e^{-nt} \sum_{k=0}^{\infty} \frac{(n^2 t)^k}{k!} (n - w)^{-k} = e^{-nt} \exp\left(\frac{n^2 t}{n - w}\right)\\
&= \exp\left(\frac{wtn}{n - w}\right), \qquad t \geq 0.
\end{aligned}$$

This shows that $(L_n^8 1)(t) = 1$ and that $(L_n^8 e^{w|u\cdot|})(t)$ converges to $e^{w|t}$ uniformly on K. By similar computations, we have $(L_n^8 u)(t) = t$. Hence the assertion follows from Theorem 2.1.

(IX). Let L_n^9 be defined for $0 \leqslant t \leqslant 1$ by

$$(L_n^9 T(\cdot))(t)x = \sum_{k=0}^{n} \binom{n}{k} t^k(1-t)^{n-k} \frac{n^k}{(k-1)!} \int_0^\infty e^{-nu} u^{k-1} T(u)x \, du. \quad (27)$$

Then L_n^9 maps $O_s(e^{w|t|}, [0, 1])$ into the set of all strongly continuous functions on $[0, 1]$.

THEOREM 3.15. *For any function $T(t)$ in $O_s(e^{w|t|}, [0, 1])$ and $x \in X$, $(L_n^9 T(\cdot))(t)\, x$ converges uniformly on $[0, 1]$ to $T(t)\, x$.*

Proof. (21) implies that for n larger than w,

$$(L_n^9 e^{w|u|})(t) = \sum_{k=0}^{n} \binom{n}{k} t^k(1-t)^{n-k} \left(\frac{n}{n-w}\right)^k = [1 - t + nt(n-w)^{-1}]^n$$
$$= [1 + wt/(n-w)]^n.$$

Differentiation with respect to w yields that $(L_n^9 u e^{w|u|})(t) = tn(n-w)^{-1}$ $[1 + wt/(n-w)]^n$. Clearly, these two identities imply that $(L_n^9 1)(t) = 1$, $(L_n^9 u)(t) = t$ and $(L_n^9 e^{w|u|})(t) \to e^{w|t|}$ uniformly on $[0, 1]$ as $n \to \infty$. Hence the assertion is true.

(X) The last example is the operator

$$(L_n^{10} T(\cdot))(t)x = \sum_{k=0}^{\infty} \binom{-n}{k} (-t)^k(1+t)^{-n-k} \frac{n^k}{(k-1)!} \int_0^\infty e^{-nu} u^{k-1} T(u)x \, du \quad (28)$$

where $t \geqslant 0$. We have

THEOREM 3.16. *Let K be any compact subset of $[0, \infty)$, $T(t) \in O_s(e^{w|t}, K)$. Then for $x \in X$, $(L_n^{10} T(\cdot))(t)\, x$ converges to $T(t)\, x$ uniformly on K as $n \to \infty$.*

Proof. Similar computations show that $(L_n^{10} 1)(t) = 1$, $(L_n^{10} u)(t) = t$, and $(L_n^{10} e^{w|u|})(t) = [1 - wt/(n-w)]^{-n}$ which converges to $e^{w|t|}$ uniformly on K. Hence the theorem follows from Theorem 2.1.

Note that the operators L_n^8, L_n^9, L_n^{10} are Szász, Bernstein and Baskakov operators with $f(k/n)$ replaced by the integral means $(n^k)/(k-1)! \int_0^\infty e^{-nu}$ $u^{k-1} f(u)\, du$.

4. Representations of Semigroups

In this section we will apply the operators studied previously to derive some representation formulas for semigroup operators.

Let $S = [T(t_1, t_2, \ldots, t_m): 0 \leqslant t_i < \infty, i = 1, 2, \ldots, m]$ be a m-parameter semigroup of bounded linear operators on X. Let us assume that S is strongly continuous on $\mathbb{R}^m_{a+} = \{t \in \mathbb{R}^m: t_i > a_i, \forall i\}$ for some a such that $a_i \geqslant 0$, $i = 1, 2, \ldots, m$. If $S_i = [T_i(t_i): 0 \leqslant t_i < \infty]$ denotes the restriction of $T(t_1, t_2, \ldots, t_m)$ to the half line $\{(0, \ldots, 0, t_i, 0, \ldots, 0): 0 \leqslant t_i < \infty\}$, then S_i is itself an one-parameter semigroup which is strongly continuous for $t_i > a_i$. S is the direct product of S_i : $T(t_1, t_2, \ldots, t_m) = \prod_{i=1}^{m} T_i(t_i)$. To each i there correspond two numbers $M_i \geqslant 1$ and $w_i \geqslant 0$ such that

$$\|T_i(t_i)\| \leqslant M_i \exp(w_i t_i), \; a_i \leqslant t_i < \infty.$$

(for this, see [4].) Hence, we have the inequality

$$\|T(t_1, t_2, \ldots, t_m)\| \leqslant M \exp(w(t_1 + t_2 + \cdots + t_m)), \qquad t \in \mathbb{R}^m_{a+},$$

where $M = M_1 M_2 \cdots M_m$ and $w = \max\{w_i : 1 \leqslant i \leqslant m\}$.

With the above properties at hand, we are prepared to derive a list of formulas. First we notice that Corollary 3.2, Theorems 3.9 and 3.13 apply to any S defined as above; Corollary 3.3, Theorems 3.5, 3.7, 3.11, and 3.12 apply to those S with $w = 0$, that is, uniformly bounded semigroups; Theorems 3.4, 3.6, 3.8, 3.10, 3.14, 3.15 and 3.16 apply to any one-parameter (C_0)-semigroup. Then, because of the nice semigroup property, namely $T_i(k_i t_i) = [T_i(t_i)]^{k_i}$, the representation formulas are easily obtained.

THEOREM 4.1. *If the m-parameter semigroup $T(t_1, t_2, \ldots, t_m)$ $(= T(t))$ of linear operators in $B(X)$ is uniformly bounded and strongly continuous on $\mathbb{R}^m_{a+}$, and if $\alpha_n/n \to 0$ as $n \to \infty$, then*

$$T(t)x = \lim_{n\to\infty} \left[I + \sum_{i=1}^{m} \left(\frac{t_i - a_i}{\alpha_n}\right)\left(T_i\left(\frac{\alpha_n}{n}\right) - I\right)\right]^n T(a)x. \tag{29}$$

holds for all $x \in X$ and for all those t which are contained in $K(\alpha_n)$ for all large n; the convergence is uniform for t in any compact set in $K(\alpha_n)$.

Note that in the case $m = 1$ or $\sup \alpha_n < \infty$, the assumption of uniform boundedness of $T(t)$ is not required. For those $T(t)$'s which are continuous in the uniform operator topology, the limit in (29) can be taken in that topology.

THEOREM 4.2. *Let $T(t)$ be as assumed in Theorem 4.1 and $\{\alpha_n\}$ be a* *sequence of positive numbers such that* $\lim_{n\to\infty} \alpha_n/n = 0$ *and* $\underline{\lim}\, \alpha_n > 0$. *For* $t \in \mathbb{R}^m_{a+}$ *and* $x \in X$ *we have*

$$T(t)x = \lim_{n\to\infty} \left[I - \sum_{i=1}^{m} \left(\frac{t_i - a_i}{\alpha_n}\right)\left(T_i\left(\frac{\alpha_n}{n}\right) - I\right)\right]^{-n} T(a)x, \tag{30}$$

uniformly for t in any compact subset of $\mathbb{R}^m_{a+}$.

Note that when $m = 1$ the assertion is true also for unbounded one-parameter semigroups. Similar conclusions as the above can be obtained for uniformly continuous semigroups.

This theorem follows from Theorem 3.5, Theorem 3.6, and the following lemma.

LEMMA. *For any bounded subset G of* $\mathbb{R}^m_{a+}$, *there is a positive integer N such that if* $t \in G$ *and* $n \geqslant N$, *then the operator*

$$A_n(t) = I - \sum_{i=1}^{m} \left(\frac{t_i - a_i}{\alpha_n}\right)\left(T_i\left(\frac{\alpha_n}{n}\right) - I\right)$$

is invertible, and

$$(A_n(t))^{-r} = \sum_k \binom{-r}{k} \prod_{i=1}^{m} \left[\frac{a_i - t_i}{\alpha_n} T_i\left(\frac{\alpha_n}{n}\right)\right]^{k_i} \left(1 - \overline{\frac{t - a}{\alpha_n}}\right)^{-r-k}, \tag{31}$$

the convergence is uniform for t in G.

Proof. The series in (31) is dominated by the series

$$\sum_k \binom{-r}{k} (-1)^k \prod_{i=1}^{m} \left(\frac{t_i - a_i}{\alpha_n}\right)^{k_i} \left\| T_i\left(\frac{\alpha_n k_i}{n}\right)\right\| \left(1 - \overline{\frac{t - a}{\alpha_n}}\right)^{-r-k}$$

$$\leqslant M \sum_{\nu=0}^{\infty} \binom{-r}{\nu} (-1)^{\nu} \left[\sum_{k=\nu} \binom{\nu}{k}\left(\frac{t_i - a_i}{\alpha_n} e^{w\epsilon_n}\right)^{k_i}\right]\left(1 + \overline{\frac{t - a}{\alpha_n}}\right)^{-r-k}$$

$$= M \sum_{\nu=0}^{\infty} \binom{-r}{\nu} (-1)^{\nu} \left[\sum_{i=1}^{m} \frac{t_i - a_i}{\alpha_n} e^{w\epsilon_n}\right]^{\nu}\left(1 + \overline{\frac{t - a}{\alpha_n}}\right)^{-r-\nu}$$

$$\leqslant M \sum_{\nu=0}^{\infty} \binom{-r}{\nu} (-1)^{\nu} \left[\frac{b_n(t)}{1 - b_n(t)} e^{w\epsilon_n}\right]^{\nu},$$

where $b_n(t) = \sum_{i=1}^{m} (t_i - a_i)/\alpha_n$ and $\epsilon_n = \alpha_n/n$. Since G is bounded and $\underline{\lim}\, \alpha_n > 0$, we can choose h such that $b_n(t) \leqslant h$ for all t in G. Hence, we have $b_n(t)\, e^{w\epsilon_n}/(1 - b_n(t)) \leqslant he^{w\epsilon_n}/(1 - h) < 1$ for all t in G and for sufficiently large n. Therefore, there is N such that the last series converges uniformly for

t in G to $M[1 - b_n(t) e^{n\epsilon_n}(1 - b_n(t))]^{-1}$ if $n \geq N$. This implies that the series in (31) converges absolutely and uniformly for t in G if $n \geq N$.

COROLLARY 4.3. *If $T(t)$ is an one-parameter semigroup strongly continuous on $[a, \infty)$, then for each $x \in X$,*

$$T(t)x = \lim_{n\to\infty} \left[2I - T\left(\frac{t-a}{n}\right)\right]^{n} T(a)x, \qquad a \leq t < \infty. \tag{32}$$

Proof. If $t = a$, this is trivial. If $t > a$, then (32) is obtained by setting $\alpha_n = t - a$ in (30).

THEOREM 4.4. *Let $T(t)$ be any m-parameter semigroup, strongly continuous on $\mathbb{R}^m_{a+}$. Then for $x \in X$, $t \in \mathbb{R}^m_{a+}$,*

$$T(t)x = \lim_{n\to\infty} \exp\left[\sum_{i=1}^{m} (t_i - a_i)n\left(T_i\left(\frac{1}{n}\right) - I\right)\right] T(a)x \tag{33}$$

Furthermore, if $T(t)$ is continuous in the uniform operator topology, then the limit holds also in that topology.

Proof. It follows from Theorem 3.8 that for each of $i = 1, 2,..., m$,

$$T_i(t_i)x = \lim_{n\to\infty} \exp\left[(t_i - a_i)n\left(T_i\left(\frac{1}{n}\right) - I\right)\right] T(a_i)x, \tag{34}$$

the convergence being uniform on any compact subset of $[a_i, \infty)$. (33) follows from (34) and the facts that $T(t) = \prod_{i=1}^{m} T_i(t_i)$, and T_i, $i = 1, 2,..., m$, commute with each other.

THEOREM 4.5. *Let $S = [T(t): t \in \mathbb{R}^m_{0+}]$ be a strongly continuous m-parameter semigroup, and A_i be the infinitesimal generator of $T_i(t_i)$, $i = 1, 2,..., m$. Then for $x \in X$, $t \in \mathbb{R}^m_{0+}$,*

$$T(t)x = \lim_{n\to\infty} \prod_{i=1}^{m} (I - n^{-1}t_i A_i)^{-n} x \tag{35}$$

uniformly for t in compact sets of $\mathbb{R}^m_{0+}$.

A similar assertion holds for uniformly continuous semigroups.

Proof. Theorem 3.10 implies that for each $i = 1, 2,..., m$,

$$T_i(t_i)x = \lim_{n\to\infty} (n/t_i)^n/(n-1)! \int_0^{\infty} \exp(-nu/t_i)\, u_i^{n-1} T(u_i)x\, du_i$$

uniformly on any compact subset of $[0, \infty)$. But the last term is equal to $(I - n^{-1}t_iA_i)^{-n}$ for all large n (see [4, p. 34]), and so (35) follows immediately.

THEOREM 4.6. *If $T(t)$ is an one parameter semigroup of class (C_0) having infinitesimal generator A, then the following representation formulas hold*:

$$T(t)x = \lim_{n\to\infty} \exp\{t[n^2(nI - A)^{-1} - nI]\}x, \qquad t \geqslant 0; \tag{36}$$

$$T(t)x = \lim_{n\to\infty} [(1 - t)I - tn(nI - A)^{-1}]^n x, \qquad 0 \leqslant t \leqslant 1; \tag{37}$$

$$T(t)x = \lim_{n\to\infty} [(1 - t)I - tn(nI - A)^{-1}]^{-n} x, \qquad t \geqslant 0. \tag{38}$$

Moreover, the convergence is uniform for t in any finite interval on which the limit holds.

Proof. Since $(n - A)^{-k} x = ((k - 1)!)^{-1} \int_0^\infty e^{-nu}u^kT(u)\, x\, du$, (see [4, p. 34]), (36), (37) and (38) follow easily from Theorem 3.14, Theorem 3.15 and Theorem 3.16, respectively.

The following two theorems are the respective versions of Theorem 3.12 and Theorem 3.13 for semigroups.

THEOREM 4.7. *Let $T(t), t \geqslant 0$, be a uniformly bounded, strongly continuous one-parameter semigroup. Then for any $x \in X$,*

$$T(t)x = \lim_{n\to\infty} \frac{t^n}{(n - 1)!} \int_0^\infty e^{-tu}u^{n-1} \left(T\left(\frac{1}{u}\right)\right)^n x\, du, \tag{39}$$

uniformly for t in any compact subset of $[0, \infty)$.

THEOREM 4.8. *Let $T(t), t \in \mathbb{R}^m$, be a strongly continuous m-parameter group of operators on X. Then for $x \in X$,* (25) *holds uniformly for t in any compact subset of $\mathbb{R}^m$.*

Remark 1. If $T(t), t \geqslant 0$, is a (C_0)-semigroup, Corollary 4.3 shows the pointwise convergence of

$$T(t)x = \lim_{n\to\infty} \left[2I - T\left(\frac{t}{n}\right)\right]^{-n} x, \qquad t \geqslant 0. \tag{40}$$

If in addition, $T(t)$ is contractive, we can deduce from Chernoff's product formula ([6] or [28]) that (40) holds uniformly for t in any compact subset of $[0, \infty)$. In fact, we can define $V(t)$ to be the function $[2I - T(t)]^{-1}, t \geqslant 0)$. Then we have $V(0) = I$, $\|V(t)\| = \|\frac{1}{2}\sum_{k=0}^\infty (T(t)/2)^k\| \leqslant \frac{1}{2}\sum_{k=0}^\infty \|T(t)/2\|^k \leqslant \frac{1}{2}\sum_{k=0}^\infty (\frac{1}{2})^k = 1$, and $V'(0)\,x = \lim_{t\to 0}((V(t) - I)/t)\,x = \lim_{t\to 0} V(t)((T(t) - I)/t)\,x = Ax$ for x in the domain $D(A)$ of the infinitesimal generator A of

$T(t)$. Thus the conditions set in Chernoff's theorem are fulfilled. and we have for any $x \in X$

$$T(t)x = \lim_{n\to\infty} \left(V\left(\frac{t}{n}\right)\right)^n x = \lim_{n\to\infty} \left[2I - T\left(\frac{t}{n}\right)\right]^{-n} x.$$

uniformly for t in any compact subset of $[0, \infty)$.

Remark 2. Theorem 4.4. with $a = 0$, gives the theorem of Dunford and Segal [9]. In the case $m = 1$, Theorem 4.1. for $a = 0$ and $x_n = 1$ reduces to Kendall's formula; Theorem 4.4 leads to Hille's first exponential formula; Theorem 4.5 gives Widder's formula. Theorem 4.6 provides new proofs for Phillips' formula (36) and the two formulas (37) and (38), due to Chung [7]. Other formulas obtained here such as the general case of (29), (30), (32) and (39) seem to be new.

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